





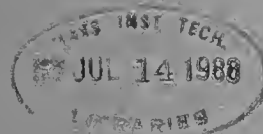
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IT TAKES  $t^*$  TO TANGO:  
TRADING COALITION IN THE EDGEWORTH PROCESS  
(Revision)

by

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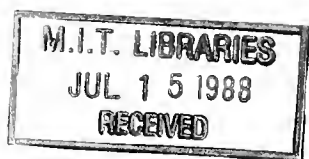
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IT TAKES  $t$  TO TANGO:

TRADING COALITIONS IN THE EDGEWORTH PROCESS

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Abstract

In the Edgeworth non-tâtonnement process, trade occurs if there exists some coalition of agents able to make a Pareto-improving trade among themselves at current prices. It is known that the size of such coalitions is bounded by the number of commodities and that, provided all agents always have strictly positive endowments, bilateral trade suffices. These results are generalized so that the maximum required coalition size is given in terms of the number of agents holding at least  $m$  commodities and the number of commodities held by at least  $k$  agents.

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Keywords: Edgeworth Process, Non-tâtonnement,

Stability of General Equilibrium,  $t$ -wise Optimality



## 1. Introduction

The basic assumption of the Edgeworth non-tâtonnement process is that trade takes place if and only if there exists a coalition of agents able to make a Pareto-improving trade among themselves at current, disequilibrium prices. Among other objections to this assumption is the possibility that it may require a very large number of agents to find each other (Fisher, 1976, p. 12, 1983, pp. 29-31). In reply to this, David Schmeidler has observed (in a private communication) that such trading coalitions need never involve more members than the number of commodities, while Paul Madden has shown that, if all agents always have strictly positive endowments of all commodities, then such coalitions need never have more than two members. (Both results can be found in Madden, 1978).

These are not very reassuring answers to the problem at hand, however, particularly if one thinks of extending the Edgeworth process to relatively realistic settings. If consumption takes place at different times, then the same commodity at different dates will be treated as different commodities. This can easily make the number of commodities much greater than the number of agents in the economy. As for Madden's bilateral trade result, it requires strictly positive endowments of all commodities for all agents, and this is far too strong a requirement in the context of disequilibrium trade.<sup>1</sup>

It is therefore of some interest to see the extent to which the two existing results can be generalized. It turns out to be possible to accomplish this with a very elementary proof, and,

while the results still do not suggest that the Edgeworth-process assumption is free of coalition-formation problems, they may have some intrinsic interest as limiting the number of traders needed for a (within trading coalition) Pareto-improving trade at given prices in a barter economy.

The results obtained depend on whether Edgeworth-process trade takes a form that I shall call "simple trade" or a form that I shall call "compound trade". Simple trade involves a circle of transactions in which each household sells one commodity and buys another (weakly) increasing its utility thereby. Compound trade involves transactions in which some household sells one commodity and buys another even though it would prefer not to do so, because the sale involved induces another household to enter into a transaction that eventually leads to an increase in the original household's utility. I shall be precise about this below and shall argue that simple trade is the natural assumption in a noncooperative, competitive setting.

When only simple trade is involved, the results are fairly rich. I show the following under very general assumptions. In a pure exchange economy, let there be  $h$  households and  $n$  commodities. For any  $m$ ,  $1 < m \leq n$ , let  $x(m)$  be the number of households holding at least  $m$  commodities in positive amounts. Then the existence of a (simple) Edgeworth-process trade implies the existence of such a trade with no more than  $t_1(m) = \text{Max} \{2, h - x(m), n - m + 2\}$  participants. In other words, the number of participants need not exceed the largest of 2, the number of households holding fewer than  $m$  commodities, and the number of commodities less  $(m - 2)$ .

Similarly, for any  $k$ ,  $1 < k \leq h$ , let  $y(k)$  be the number of commodities held in positive amounts by at least  $k$  households. Then the existence of a (simple) Edgeworth-process trade implies the existence of such a trade with no more than  $t_2(k) = \text{Max } \{2, n - y(k), h - k + 2\}$  participants. In other words, the number of participants need not exceed the largest of 2, the number of commodities held by fewer than  $k$  households and the number of households less  $(k - 2)$ .

Putting these results together, Edgeworth-process coalition size need not exceed  $t^* = \text{Min}_m \{ \text{Min}_m t_1(m), \text{Min}_k t_2(k) \}$ . Evidently, if  $t^* > 2$ , then  $t^*$  is no larger than the number of households who do not hold a positive stock of all commodities or the number of commodities not held in positive amounts by all households. There are other results as well.

When trade can be compound, however, fewer results are available. Here the only new result is that coalition size need not exceed the larger of 2 and the number of households not holding all commodities.

## 2. The $t$ -wise Optimality Literature

These results generalize and strengthen those of Schmeidler and Madden. They are, however, considerably weaker than results on the related question of when " $t$ -wise optimality" -- the non-existence of Pareto-improving trades involving no more than  $t$  traders for some arbitrary  $t$  -- is equivalent to full Pareto optimality. (See Feldman, 1973, Graham, Jennergen, Peterson, and Weintraub, 1976, Madden, 1975, Rader, 1968, 1976, and, especially, Goldman and Starr, 1982.) The reasons for that difference

are instructive and are best understood after an example.

Consider Theorem 1.1 in Goldman and Starr (1982, p. 597), a theorem originally due to Rader. It states that, provided there is a trader holding positive quantities of all goods, then the absence of any mutually-improving bilateral trade implies Pareto-optimality, so that there are no mutually-improving trades for any number of traders. Put differently, the existence of some mutually-improving trade implies the existence of a mutually-improving bilateral trade.

The proof of this theorem consists in observing that prices corresponding to the marginal utilities of the trader who holds all goods (say trader 1) must support a Pareto optimum since otherwise some mutually-improving bilateral trade would be possible. True enough. If, for trader 2, the marginal rate of substitution between some pair of goods were different than it is for trader 1, then a mutually-improving bilateral trade between them would be possible at some other set of prices.

Note, however, that this leads to a contradiction only because it is assumed that no mutually-improving bilateral trade is possible at any prices. The equivalent assumption in the present case would be the much weaker one that no mutually-improving bilateral trade is possible at a given set of prices. That this does not lead to the same result can be seen by observing that, if prices happen to be equal to trader 1's marginal utilities, trader 1 will not wish to trade. Hence, the possible trade between traders 1 and 2 will not be possible at the given prices, and no contradiction arises. Indeed, in this situation

(without further assumptions), there is nothing to prevent there from being a mutually-improving trade involving several (or all) traders other than trader 1.

The general point is as follows. In showing that the existence of some mutually-improving trade implies the existence of such a trade with no more than  $t$  traders, the  $t$ -wise optimality literature effectively considers the case in which no  $t$ -wise improving trade is possible at any set of prices. This is a much stronger assumption than the condition that no  $t$ -wise improving trade be possible at given prices, and it is therefore not surprising that it leads to much stronger results.

Since trade in actual economies often takes place at given prices, however, it is interesting to know how many traders are required with prices fixed.

### 3. Preliminaries: Simple Trades and Compound Trades

There are  $h$  households and  $n$  commodities. Each household has a differentiable, locally-nonsatiated, strictly quasi-concave utility function that is non-decreasing in its arguments.<sup>2</sup> Prices are assumed to be strictly positive. (This is mainly a convenience.<sup>3</sup>)

Definition 1. An Edgeworth-process trade is a trade at given prices such that, with all participants in the trade on their budget constraints, no participant's utility decreases and at least one participant's utility increases.

For later purposes, observe that the strict quasi-concavity of the utility functions implies that an Edgeworth-process trade remains an Edgeworth-process trade if the amounts of each commo-

dity traded by each participant are all multiplied by the same scalar,  $\lambda$ ,  $0 \leq \lambda \leq 1$ . We can thus consider very small trades and work in terms of marginal rates of substitution.

Lemma 1. In an Edgeworth-process trade, it is not possible to partition the participating households into two sets, A and B, such that some household in A sells some commodity to a household in B but no household in B sells any commodity to any household in A.

**Proof.** Suppose not. Then the total wealth of households in A would be greater after the trade than before, contradicting the fact that all households must remain on their budget constraints.

Now consider an Edgeworth-process trade which involves household  $i$  selling commodity  $j$  to household  $i'$ . Household  $i'$  must sell to some other household or households, and they in turn must sell to others, and so on. All of these households will be said to buy commodity  $j$  from household  $i$ , directly or indirectly. Denote the set of such households as  $B(i, j)$ .

Lemma 2. Household  $i$  is a member of  $B(i, j)$ .

**Proof.** Suppose not. Take A as the set of households involved in the Edgeworth-process trade that are not in  $B(i, j)$ . Take B as  $B(i, j)$ . Then Lemma 1 is contradicted.

Hence every sale (or purchase) of a commodity by a household in an Edgeworth-process trade involves a circle of households and commodities, with each household in a circle buying a commodity from the preceding one and selling a commodity to the succeeding one. We can think of transactions in which a given household sells more than one commodity to another as involving more than



one circle (possibly with all the same households and almost all the same commodities).

Definition 2. An Edgeworth-process trade is called "simple" if at least one of the circles composing it is itself an Edgeworth-process trade. An Edgeworth-process trade that is not simple will be called "compound".

In other words, in a simple Edgeworth-process trade, the households participating in at least one circle would be willing to do so even if they were not also participating in other circles. In a compound Edgeworth-process trade, on the other hand, at least one household participating in any circle only does so because such participation is required to bring a different circle into existence.

An example will help here. Figure 1 shows a trade consisting of two "circles". Nodes in the diagram represent households, indicated by numbers, while arrows denote sales of commodities, indicated by letters. Thus, in the diagrammed trade, the right-hand "circle" has household 1 selling commodity a to household 2, household 2 selling commodity b to household 3, and household 3 selling commodity c to household 1. In the left-hand "circle", household 1 sells commodity d to household 4, household 4 sells commodity e to household 3, and household 3 sells commodity c to household 1.

This trade would be simple if at least one of these "circles" were (weakly) utility-improving for all its participants. But suppose that the situation is as follows. At the prices at which trade takes place, households 2 and 4 find their respective

roles in the diagrammed trade to be utility increasing. Household 1, however, would not be willing to participate in the left-hand circle standing alone. That is, at the given prices, household 1 would not be willing to sell d and buy c. It would, on the other hand, be happy to engage in the right-hand circle standing alone, selling a and buying c. By contrast, household 3 would be willing to participate in the left-hand circle standing alone (selling c and buying e), but would not be willing so to participate in the right-hand one (selling c and buying b). In this circumstance, neither circle, standing alone, would be an Edgeworth-process trade. Nevertheless, the entire transaction taken as a whole can be an Edgeworth-process trade, with household 1 agreeing to participate in the left-hand circle in exchange for household 3's agreement to participate in the right-hand one.

Without the strong assumption that all commodities are held in positive amounts by all households, there is nothing to prevent the possibility that the only utility-improving trades possible at given prices are compound.<sup>4</sup> The assumption that such trades will nevertheless take place seems very strong, however, and somewhat out of place in a non-cooperative, competitive setting.

Where only utility-improving circle trades are involved, one can imagine prices being announced and each household then listing the commodities that it would like to buy and that it would like to sell. Someone (the "market") then arranges (small) circle trades accordingly. If each household understands that any one of its commitments to purchase may be paired with any one

of its commitments to sell, whether or not any other pairing takes place, then offers will be made in such a way that any (small enough) utility-improving trade is simple. The construction of compound trades, on the other hand, requires more information. At the very least, it requires households to specify a preference ordering for goods, in the sense that (at the given prices) the household is willing to exchange any good lower in the ordering for any good higher up (at least in small amounts).

To put it another way, the construction of a compound trade requires considerably more information about preferences than the construction of a simple one. This is particularly so for trades more complex than that of Figure 1 in which many more than two of the households participate in one or more circles as the guid pro g u g for obtaining participation in another one.

In the present state of Edgeworth-process analysis, this does not matter. With stability proofs depending on positive commodity holdings by all participants, it is not necessary to construct compound trades. If that unreasonably strong assumption is ever to be relaxed, however, it would be very desirable to have a proof of Edgeworth-process stability that assumes only that trade takes place if simple utility-improving trade is possible and does not require participants to find compound trades.

In any event, in or out of the specific Edgeworth-process context, it is obviously interesting separately to consider simple and compound trades, and I shall do so.

### 3. Standard t-trades

I begin with simple trades. If an Edgeworth-process trade is simple, then at least one of the circles of which it is composed is itself an Edgeworth-process trade. Hence, in considering the maximum number of participants required for a simple Edgeworth-process trade, it suffices to assume that the trade involved is itself just a single circle.

Furthermore, if a given commodity occurs twice in such a circle, then the number of participants in the circle can be reduced. Consider the trade diagrammed in Figure 2. Here, a, b, c, and d are all different commodities. Suppose that commodity x is the same as any one of the other four commodities. If  $x = a$ , then household 1 can be removed from the trade. If  $x = b$ , then households 1 and 2 can be removed. If  $x = c$ , then households 1, 2, and 3 can all be removed, making bilateral trade possible. Finally,  $x = d$  is impossible if household 5 gains from trade, and, in any case,  $x = d$  implies that household 5 can be removed from the trade. There is nothing special about this example.

It follows that, in considering simple Edgeworth-process trades, it suffices to look at circles with the same number of commodities as households. It will be convenient to standardize notation as follows.

Definition 3. A standard t-trade is a circle of households, which we may as well take to be  $\{1, \dots, t\}$ , and a set of commodities, which we may as well take to be also  $\{1, \dots, t\}$ , such that, for  $1 \leq i < t$ , household i sells commodity i to household i+1, while household t sells commodity t to household 1.

I shall adopt the convention that, when considering a standard  $t$ -trade, commodity 0 is taken to be commodity  $t$ , so that each household  $i = 1, \dots, t$  sells commodity  $t$  and buys commodity  $t-1$ . I denote  $\{i-1, i\}$  by  $S(i)$ .

#### 4. Simple Trades: Results

The following fairly obvious fact is central to the analysis of simple trades.

**Lemma 3.** Consider any household,  $H$ , and any triplet of commodities,  $a, b, c$ , with  $H$ 's holdings of  $a$  and  $b$  both positive. Suppose that, at current prices,  $H$  could increase utility by selling  $a$  and buying  $c$ . Then, at the same prices,  $H$  would also find one of the following trades to be utility-increasing: (1) selling  $b$  and buying  $c$  or (2) selling  $a$  and buying  $b$ .

**Proof.** As before, denote  $H$ 's utility function by  $U(\cdot)$ . Let the prices of the three goods be  $p_a$ ,  $p_b$ , and  $p_c$ , respectively. Then  $U_a/U_c < p_a/p_c$ , since  $H$  could increase utility by selling  $a$  and buying  $c$ . Evidently, either  $U_b/U_c < p_b/p_c$ , in which case  $H$  would find selling  $b$  and buying  $c$  to be utility increasing, or else  $U_a/U_b < p_a/p_b$ , in which case  $H$  would find selling  $a$  and buying  $b$  to be utility increasing.

This leads to the following lemma from which almost all later results are derived.

**Lemma 4.** Suppose that an Edgeworth-process standard  $t$ -trade is possible with  $t > 2$ . Suppose further that household  $i$  ( $1 \leq i \leq t$ ) holds a positive amount of some commodity  $j$  ( $1 \leq j \leq t$ ), with  $j$  not in  $S(i)$ . Then there is an Edgeworth-process trade

involving no more than  $t-1$  households.

**Proof.** Without loss of generality, we can take  $i = 1$ . Then household 1, which certainly holds commodity 1 also holds commodity  $j$ , where  $1 < j < t$ . By Lemma 3, either household 1 is willing to sell commodity 1 and buy commodity  $j$  or else it is willing to sell commodity  $j$  and buy commodity  $t$ .

Suppose first that household 1 is willing to sell commodity 1 and buy commodity  $j$ . Then there is a standard  $j$ -trade possible. That is, households  $\{1, \dots, j\}$  can trade with each household,  $g$ , selling commodity  $g$  to household  $g+1$  and household  $j$  selling commodity  $j$  to household 1. Since  $j < t$ , there are at most  $t-1$  households involved in this trade.

Now suppose that household 1 is willing to sell commodity  $j$  and buy commodity  $t$ . In this case, households  $\{j+1, \dots, t, 1\}$  can trade with household  $g$  selling commodity  $g$  to household  $g+1$ , except that household  $t$  sells commodity  $t$  to household 1, and household 1 sells commodity  $j$  to household  $j+1$ . The number of households involved in this trade is  $(t + 1 - j)$ , and this is less than  $t$ , since  $j > 1$ .

It is now easy to prove the main result for simple trades:

**Theorem 1.** (A) For any  $m$ ,  $1 < m \leq n$ , let  $x(m)$  be the number of households holding at least  $m$  commodities in positive amounts. If there exists a simple Edgeworth-process trade, then there exists one with at most  $t_1(m) = \max \{h - x(m), n - m + 2\}$  participants.

(B) For any  $k$ ,  $1 < k \leq h$ , let  $y(k)$  be the number of commodities held by at least  $k$  households in positive amounts. If there

exists a simple Edgeworth-process trade, then there exists one with at most  $t_2(k) = \text{Max} \{n - y(k), h - k + 2\}$  participants.

**Proof.** (A) Without loss of generality, suppose that there exists an Edgeworth-process standard  $t$ -trade with  $t > t_1(m)$ . Since  $t > h - x(m)$ , at least one of the households involved in the trade must hold at least  $m$  goods. Let that household be household  $i$ . Then  $i$  holds at least  $(m - 2)$  goods not in  $S(i)$ . Since  $t > n - (m - 2)$ ,  $i$  must hold some good involved in the trade that is not in  $S(i)$ . Since  $t > n - m + 2 \geq 2$ , Lemma 4 now yields the desired result.

(B) Again suppose that there exists an Edgeworth-process standard  $t$ -trade with  $t > t_2(k)$ . Since  $t > n - y(k)$ , at least one of the commodities involved in the trade is held by at least  $k$  households. Let that commodity be commodity  $j$ . Then  $j$  is held by at least  $(k - 2)$  households,  $i$ , with  $j$  not in  $S(i)$ . Since  $t > h - (k - 2)$ , at least one such household must be involved in the trade. Since  $t > h - k + 2 \geq 2$ , the desired result again follows from Lemma 4.

Corollary 1. If there exists a simple Edgeworth-process trade, then there exists one with at most

$$t^* = \text{Min} \left\{ \text{Min}_m t_1(m), \text{Min}_k t_2(k) \right\}$$

participants (where the notation is as in Theorem 1).

**Proof.** Obvious.

Corollary 2. If a simple Edgeworth-process trade exists, then one exists with no more than  $\text{Max} \{2, \text{Min} (h - x(n), n - y(h))\}$  participants.

participants.

Proof. Set  $m = n$  and  $k = h$  in Theorem 1.

Corollary 2 states that, if a simple Edgeworth-process trade requires more than two participants, it need not require more than the number of households not holding all commodities or the number of commodities not held by all households.

Corollary 3 (Schmeidler). If a simple Edgeworth-process trade exists, then one exists with no more than  $n$  participants.

Proof. Follows from Corollary 2 and the fact that  $y(h) \geq 0$ .

Corollary 4. Suppose that at least  $h-2$  households hold  $m \geq 2$  commodities (not necessarily the same ones). Then, if a simple Edgeworth-process trade exists, such a trade exists with no more than  $n-m+2$  participants.

Proof. In Theorem 1 (A),  $x(m) \geq h-2$ .

Corollary 5. Suppose that at least  $n-2$  commodities are held by  $k \geq 2$  households (not necessarily the same ones). Then, if a simple Edgeworth-process trade exists, such a trade exists with no more than  $h-k+2$  participants.

Proof. In Theorem 1 (B),  $y(k) \geq n-2$ .

These results obviously imply:

Corollary 6. Suppose that either (a) at least  $h-2$  households hold all commodities in positive amounts or (b) at least  $n-2$  commodities are held in positive amounts by all households. If a simple Edgeworth-process trade exists, then a bilateral Edgeworth-process trade exists.



This is a slightly stronger version of:

Corollary 7 (Madden). Suppose that all households hold positive amounts of all commodities. If a simple Edgeworth-process trade exists, then a bilateral Edgeworth-process trade exists.

### 5. Simple Trades: Can Further Results Be Obtained?

The number of participants required for an Edgeworth-process trade depends on the distribution of commodity holdings and, of course, on the distribution of tastes. The results so far obtained for simple trades have made no assumptions on the distribution of tastes and have only characterized the distribution of commodity holdings by the two functions,  $x(\cdot)$  and  $y(\cdot)$ , (respectively, the number of households holding at least a given number of commodities and the number of commodities held by at least a given number of households).

Since that information does not completely characterize the holding of commodities by households, it is easy to see that more information on the pattern of such holdings can make a considerable difference. To see this, consider the following example:

(A) Assume  $h = n > 2$ , with  $n$  even. Suppose that there exists an Edgeworth-process standard  $n$ -trade with household  $i$  holding only the commodities in  $S(i)$  (that is, commodities  $i-1$  and  $i$ , with commodity 0 taken to be commodity  $n$ ). Then  $x(2) = h$ , while  $x(m) = 0$  for  $m > 2$ . Similarly,  $y(2) = n$ , while  $y(k) = 0$  for  $k > 2$ . This means that  $t_1(m) \equiv n \equiv t_2(k)$  for  $1 < m \leq n$  and  $1 < k \leq h$ . Evidently,  $t^* = n$  in Corollary 1, and, indeed, it is obvious that the standard  $n$ -trade cannot be reduced.

(B) With the same number of goods and households as in (A),

now holds only the commodities  $i$  and  $i+1$ , instead of  $i$  and  $i-1$  (with commodity  $n+1$  taken to be commodity 1). Then the functions  $x(\cdot)$  and  $y(\cdot)$  are the same as in (A), so that  $t^* = n$ , as before. In this case, however, every household,  $i$ , owns a good involved in the standard  $n$ -trade that is not in  $S(i)$ , so that Lemma 4 shows the existence of an Edgeworth-process trade with fewer than  $n$  participants. In fact, it is not hard to show that there exists such a trade with  $n/2$  participants, since, along the lines of the proof of Lemma 4, every odd-numbered participant in the standard  $n$ -trade can bypass participant  $i+1$ .

Somewhat more surprising than this is the fact that  $t^*$  of Corollary 1 need not be the least upper bound on required trades given only the information in  $x(\cdot)$  and  $y(\cdot)$ . To see this, consider the following example.

Suppose that there exists an  $m^* \geq 3$ , with  $n - m^* + 2 > m^*$ , such that:

$$(1) \quad x(m) = \begin{cases} n & \text{for } m \leq m^* \\ 0 & \text{for } m > m^* \end{cases} ; \quad y(k) = \begin{cases} n & \text{for } k \leq m^* \\ 0 & \text{for } k > m^* \end{cases}.$$

In other words, every household owns exactly  $m^*$  commodities, and every commodity is owned by exactly  $m^*$  households. Assume  $n \geq n$ . In this case,

$$(2) \quad t_1(m) = \begin{cases} n - m + 2 & \text{for } m \leq m^* \\ n & \text{for } m > m^* \end{cases},$$

and, similarly,

$$(3) \quad t_2(k) = \begin{cases} n - k + 2 & \text{for } k \leq m^* \\ n & \text{for } k > m^* \end{cases}.$$

Then  $t^* = t_1(m^*) = n - m^* + 2$ . Note that, by assumption,  $t^* > m^*$ , so that  $y(t^*) = 0$ . In other words, there is no commodity owned by as many as  $t^*$  households.

Now consider any simple Edgeworth-process trade involving  $t^*$  households. Without loss of generality, we may take this to be the standard  $t^*$ -trade. There are  $n - m^* + 2$  goods involved in such a trade. Each household,  $i$ , owns at least  $(m^* - 2)$  goods not in  $S(i)$ . In order for none of these goods to be involved in the trade, those  $(m^* - 2)$  goods must be the same for all  $t^*$  participants. Since we know that this is impossible, Lemma 4 tells us that there is an Edgeworth-process trade with fewer than  $t^*$  participants.

It remains to show that the functions  $x(\cdot)$  and  $y(\cdot)$  given in (1) can actually occur. This is easily done by having household  $i$  own commodities  $\{i-1, i, \dots, i+m^*-2\}$ , with commodity 0 identified with commodity  $n$  and commodity  $n+j$  identified with commodity  $j$  ( $1 \leq j \leq m^*-2$ ).

Evidently, further work along these lines can produce stronger lower bounds on the number of required participants than  $t^*$ . That further analysis would have to run in terms of the total number of commodities owned by sets of the households and the total number of households that own sets of commodities. Since requirements even on the number of commodities held by a single household already strain what we can reasonably assume about a disequilibrium process, I do not believe that such further results will be worth the complications involved. I am not certain

of this, however, and the analysis of this problem may be a suitable subject for further research.

### 6. Compound Trades: Results

I now turn to the more complex case of compound trades. Here results are not so easily come by. The principal reason for this is that the result of Lemma 4 does not hold for compound trades. To see this, consider Figure 1 again and recall that household 1 participates in the left-hand circle only in order to participate in the right-hand one, while the opposite is true for household 3. Suppose that household 4 owns commodity  $c$ . Then, by Lemma 3, household 4 is either willing to sell  $c$  and buy  $d$  or else willing to buy  $c$  and sell  $e$ . In the latter case, bilateral trade between households 3 and 4 is possible, but suppose that the former case applies, and that household 4 has no interest in purchasing either  $a$  or  $b$ . If the left-hand circle were itself an Edgeworth-process trade, bilateral trade between households 1 and 4 would be possible, but now it is not. Household 1 is not willing merely to sell  $d$  and buy  $c$ ; it is doing so only because that gives it the opportunity to sell  $a$  and buy  $c$ . If we try to replace the left-hand circle by a bilateral trade between households 1 and 4, household 3 will no longer receive  $e$ . Since household 3's participation in the right-hand circle is conditional on its getting  $e$ , household 3 will no longer participate in the right-hand circle. But, in that case, household 1 will have no reason to sell  $d$  and buy  $c$ , and the whole trade will break down.

Moreover, it is not true (as it is in the case of simple

trades) that the existence of a commodity involved in the trade and held by all participants implies that the number of participants can be reduced. To see this, consider Figure 3. Here there are two circles each involving the same three households. (Household 2 has been exhibited twice for clarity.) Suppose that household 2 gains utility from participation in the right-hand circle and loses from participation in the left-hand one, while the opposite is true for households 1 and 3. Assume that household 1 owns a and d, household 2 owns a, b, and e, and household 3 owns a and c. Thus, a is owned by all households.

For convenience, assume that all prices are equal to unity. Denoting the utility function of household  $i$  by  $U^i$  and marginal utilities by subscripts, the information given implies:

$$(4) \quad U_a^1 > U_c^1 > U_d^1 ,$$

$$(5) \quad U_a^2 > U_b^2 ; \quad U_d^2 < U_e^2 ,$$

and

$$(6) \quad U_e^3 > U_c^3 > U_b^3 .$$

Suppose that, in addition,

$$(7) \quad U_d^1 > U_b^1 > U_e^1 ,$$

$$(8) \quad U_a^2 > U_d^2 ; \quad U_a^2 > U_c^2 ,$$

and

$$(9) \quad U_a^3 > U_e^3 ; \quad U_b^3 > U_d^3 .$$

Inequalities (4) and (7) imply that household 1 will not

sell either a or d in order to purchase b or e, while (8) implies that household 2 will not sell a in order to purchase d. Hence bilateral trade between households 1 and 2 cannot take place.

Similarly, inequalities (7) and (9) imply that household 3 will not sell a to purchase b or e, while (8) implies that household 2 will not sell a in order to purchase c. Hence bilateral trade between households 2 and 3 cannot take place.

Finally, (4) implies that household 1 will not sell a in order to purchase c, while (6) and (9) imply that household 3 will not sell a in order to purchase d. Hence bilateral trade between households 1 and 3 cannot take place.

Thus, no bilateral trade is possible, and the compound trade shown in Figure 2 cannot be reduced even though a is owned by all participants. Basically, the reasoning that led to a different result in the case of simple trades breaks down (as before) because the trades are all interdependent. Thus, the household that owns a good that it is not trading is household 3. That household would be glad to purchase a and sell c, and, if the right-hand circle were itself an Edgeworth-process trade, this would allow household 3 to replace household 2 and deal directly with household 1. In the compound case being examined, however, households 1 and 3 have no direct interest in such a trade of a for c, and engaging in it would remove household 2's reason for participating in the left-hand circle.

This does not mean that it is impossible to obtain positive results, however. In fact, the parallelism between commodities and households breaks down in the case of compound trades, for it

remains true that the presence of a household owning all goods in a multilateral trade permits a reduction in the number of participants. To see this, consider the following lemma which gives the result parallel to (and weaker than) that of Lemma 4 for the case of compound trades.

Lemma 5. Suppose that an Edgeworth-process trade exists with  $t > 2$  participants. Suppose further that there exist two households,  $i$  and  $i'$ , participating in the trade, such that the set of commodities owned in positive amounts by household  $i$  includes all commodities being traded (bought or sold) by household  $i'$ . Then there exists an Edgeworth-process trade with no more than  $t-1$  participants.

Proof. Household  $i'$  can be thought of as buying one composite good (a linear combination of ordinary goods) and selling another. (For example, household 1 in Figure 1 buys  $c$  and sells a combination of  $a$  and  $d$ .) Let  $B$  denote the composite good that household  $i'$  buys, and  $S$  the composite good that it sells. Then household  $i$  owns both  $B$  and  $S$ . If household  $i$  would find it strictly utility increasing to sell  $B$  and buy  $S$ , then a bilateral Edgeworth-process trade between households  $i$  and  $i'$  is possible. If, on the other hand, household  $i$  would not find it strictly utility increasing to sell  $B$  and buy  $S$ , then household  $i$  can replace household  $i'$  in the original trade, selling  $S$  and buying  $B$ .

This leads immediately to the extension of (parts of) Corollaries 2, 6, and 7, above, to the case of compound trades. (As before,  $h$  is the number of households,  $n$  the number of commodi-

ties, and  $x(n)$  the number of households holding positive amounts of all commodities.)

Theorem 2. If an Edgeworth-process trade exists, then one exists with no more than  $\text{Max}\{2, h - x(n)\}$  participants.

Proof. Obvious from Lemma 5.

Corollary 8. Suppose that at least  $h-2$  households hold all commodities in positive amounts. If an Edgeworth-process trade exists, then a bilateral Edgeworth-process trade exists.

This is a slightly stronger version of:

Corollary 9 (Madden). Suppose that all households hold positive amounts of all commodities. If an Edgeworth-process trade exists, then a bilateral Edgeworth-process trade exists.<sup>5</sup>



## NOTES

\* I am indebted to a referee for comments and to Peter A. Diamond for helpful discussion but retain responsibility for error. I wish to dedicate this paper to the memory of my aunt and dancing teacher, Ethel Fisher Korn.

1. Note, however, that existing proofs of stability in the Edgeworth process require the positive endowment assumption anyway (Hahn, 1962, Uzawa, 1962, Arrow and Hahn, pp. 328-337). Thus Madden's result formally answers the criticism that large numbers of traders may be required. That answer will not be satisfactory, however, if the analysis is ever to be advanced beyond such a strong assumption.

2. Strict quasi-concavity is not to be interpreted to rule out the possibility of satiation in one or more (but not all) goods, so that indifference surfaces can become parallel to one or more of the axes.

The assumption of differentiability can almost certainly be weakened to the requirement that indifference surfaces have unique supporting hyperplanes (Madden, 1978, p. 281), but there seems little gain in complicating the exposition to do so. Apart from the method of proof used, one needs to rule out cases such as the following. Suppose that household 1 regards apples and bananas as perfect complements while households 2 and 3 do not. In that circumstance, the three households may have a Pareto-improving trade in which 1 sells carrots to 2 for apples and to 3 for bananas. Such a trade can require three participants even though a particular household (1) participates in all transactions. This makes calculation of the minimum number of partici-

pants tedious at best, and, as the circumstance involved is quite special, it does not seem worth pursuing (although it might be possible to handle it along the lines of the treatment given to "compound" trades below). (Note that if all agents view a given subset of commodities as perfect complements using the same proportions, then, without loss of generality, that subset can be renamed as a composite commodity.)

3. The principal complication avoided is that of keeping track of gifts in which one household gives a free good to another without getting anything in return.

4. It is easy to see, however, that the existence of a bilateral Edgeworth-process trade implies the existence of a simple bilateral Edgeworth-process trade.

5. Schmeidler's result, while true, does not seem readily provable for compound trades along the lines here developed.

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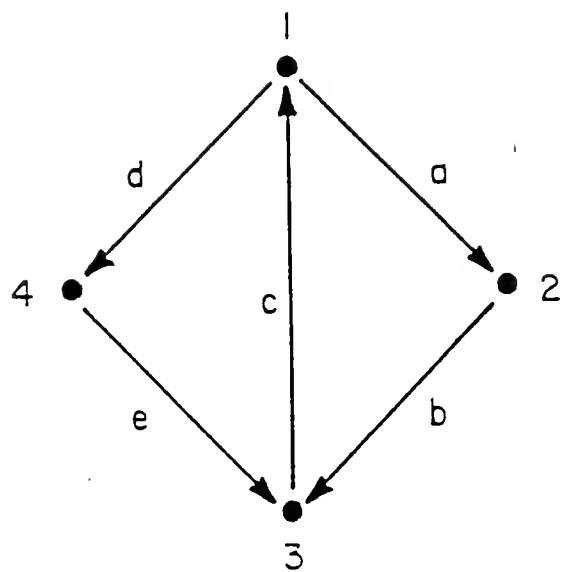


FIGURE 1



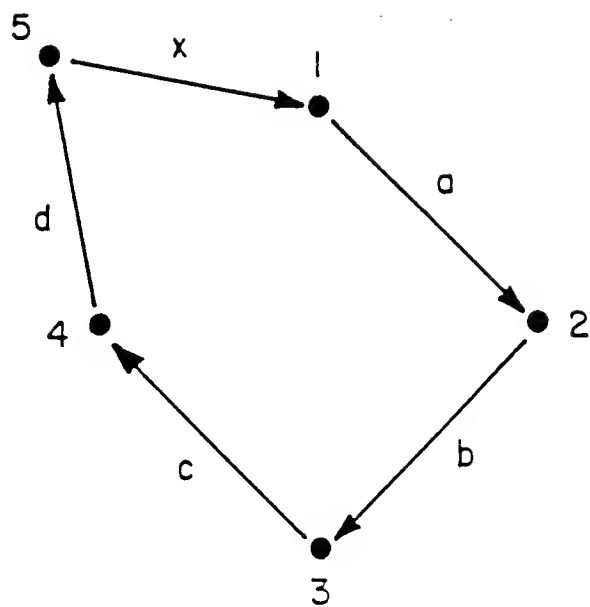


FIGURE 2





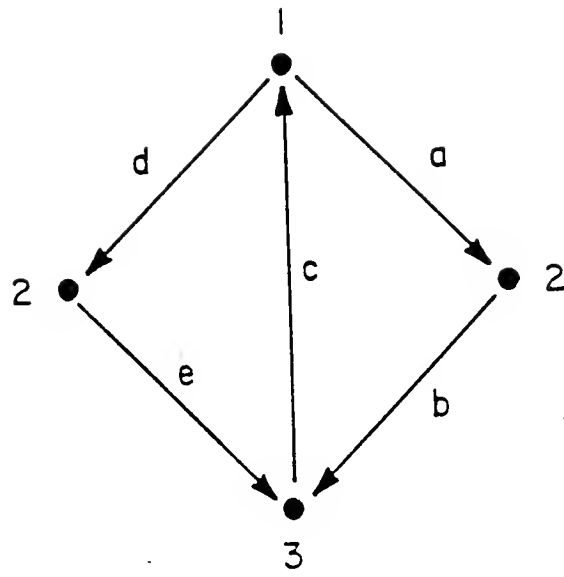
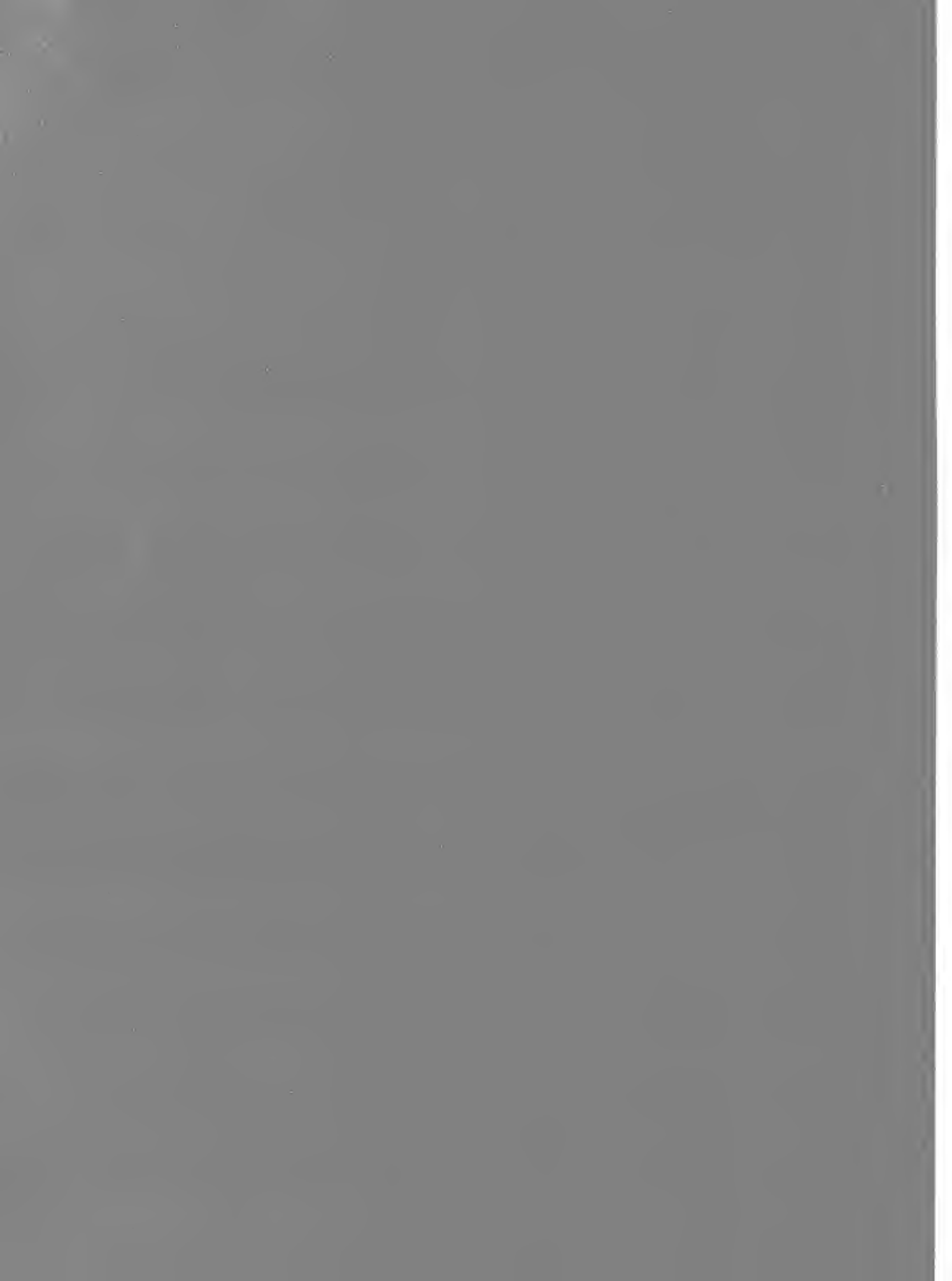


FIGURE 3

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